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The $0^+ \to 0^+$ positron double-β decay with emission of two neutrinos in the nuclei ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd

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Abstract. Theoretical results for two neutrinos in the nuclei ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd are presented. The study employs the Hartree-Fock-Bogoliubov model to obtain the wave functions of the parent and daughter nuclei, in conjunction with the summation method to estimate the double-beta decay nuclear matrix elements. The reliability of the intrinsic wave functions of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei is tested by comparing the theoretically calculated spectroscopic properties with the available experimental data. The calculated half-lives $T^{2\nu}_{1/2}$ of ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd nuclei for 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν $ECEC$ modes are presented. The effect of deformation on the nuclear transition matrix element $M_{2\nu}$ is also studied.

PACS. 23.40.Hc Relation with nuclear matrix elements and nuclear structure – 21.60.Jz Hartree-Fock and random-phase approximations – 23.20.-g Electromagnetic transitions – 27.60.+j $90 \leq A \leq 149$

1 Introduction

The two-neutrino double-beta (2ν $\beta\beta$) decay and the neutrinoless double-beta (0ν $\beta\beta$) decay can occur in four different processes: double-electron ($\beta^-\beta^-$) emission, double-positron ($\beta^+\beta^+$) emission, electron-positron conversion (β^+EC) and double-electron capture ($ECEC$). The later three processes are energetically competing and we shall refer to them as positron double-beta decay (e^+DBD) modes. The 2ν $\beta^-\beta^-$ decay is allowed in the standard model of electroweak unification (SM) and the half-life of this process has been already measured for about ten nuclei out of 35 possible candidates. Hence, the absolute values of the nuclear transition matrix elements (NTMEs) $M_{2\nu}$ can be extracted directly. Consequently, the validity of different models employed for nuclear-structure calculations can be tested by calculating the $M_{2\nu}$. In case of 2ν e^+DBD modes, experimental limits on half-lives have already been given for 14 out of 34 possible isotopes. The observation of 2ν e^+DBD modes would further constrain the nuclear models employed to study the $\beta\beta$ decay severely.

On the other hand, the 0ν $\beta\beta$ decay violates the lepton number conservation and is possible in gauge theoretical models beyond the SM as GUTs, Majoron models, R_p violating SUSY models, lepto quark exchange and compositeness scenario. The aim of all the present experimental activities is to observe the 0ν $\beta\beta$ decay. The observation of 0ν e^+DBD modes would play a crucial role in discriminating finer issues like the dominance of Majorana neutrino mass or right-handed currents. The experimental aspects and theoretical implications of e^+DBD modes have been widely discussed over the past years [1–11].

The experimental study of $\beta^-\beta^-$ decay is usually preferable due to a larger available phase space in comparison to e^+DBD modes. On the other hand, the e^+DBD modes are attractive from the experimental point of view due to the fact that they can be easily separated from the background contaminations and easily detected through coincidence signals from four γ-rays, two γ-rays and one γ-ray for $\beta^+\beta^+$, β^+EC and $ECEC$ modes, respectively. In the case of the 2ν $ECEC$ mode, the Q-value of ^{106}Cd is pretty large, 2.782 MeV, but the detection of the $0^+ \to 0^+$ transition is difficult since only X-rays are emitted.

In 1955, Winter studied the e^+DBD modes of ^{106}Cd experimentally to explore the possibility of distinguishing between the Dirac or Majorana character of the electron neutrino [12]. The 2ν e^+DBD modes were studied theoretically for the first time by Rosen and Primakoff [1]. Following the discovery of parity violation in beta decay, there was a marked decline in the experimental searches of

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$\beta\beta$ decay in general as both the lepton number conservation and the γ_5 invariance had to be violated for the 0ν $\beta\beta$ decay to occur. However, the perception began to change after Vergados showed that e^+DBD modes are possible as lepton number violating process in gauge theories beyond the SM [2]. Kim and Kubodera estimated the half-lives of all the three modes with modified NTMEs and non-relativistic phase space factors [13]. Abad *et al.* performed similar calculations using relativistic Coulomb wave functions [14]. Some other theoretical studies were also done for the e^+DBD modes [15–18]. The experimental activities on the study of 2ν e^+DBD modes were also resumed [19–21]. In the meantime, the QRPA emerged as a successful model in explaining the quenching of NTMEs by incorporating the particle-particle part of the effective nucleon-nucleon interaction in the proton-neutron channel [22] and the observed $T_{1/2}^{2\nu}$ of several 2ν $\beta^-\beta^-$ decay emitters were reproduced successfully [7]. Subsequently, the 2ν e^+DBD modes were studied in shell model, QRPA and its extensions, $SU(4)_{\sigma\tau}$ and SSDH and pseudo $SU(3)$ [7].

Low-background set-ups using Ge detectors were proposed by Barabash [23] to detect the transition of 2ν $ECEC$ mode to the 0_1^+ excited state. New developments in experimental set-ups have led to good limits on the measurement of the 2ν e^+DBD modes of nuclei of our interest namely ^{106}Cd [12,20,24–30] and ^{108}Cd [24,28,29] through the direct counting experiments. In the mass region $A \sim 100$, Norman has studied the 2ν e^+DBD modes of ^{96}Ru [21] and ^{102}Pd is also a potential candidate to be studied with a Q-value of about 1.175 MeV with natural abundance of about 1.02%. With improved sensitivity in detection systems of the planned bigger Osaka-OTO experiment [31] and COBRA [32], it is expected that 2ν e^+DBD modes will be in observable range in the near future. Hence, a timely reliable prediction of the half-lives of ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd nuclei will be helpful in the ongoing planning of future experimental set-ups.

The structure of nuclei in the mass region $A \approx 100$ is quite complex. This mass region offers a nice example of shape transitions, *i.e.* sudden onset of deformation at neutron number $N = 60$. The nuclei are soft vibrators for $N < 60$ and quasi-rotors for $N > 60$. The nuclei with neutron number $N = 60$ are transitional nuclei. In this mass region, $A = 96$–108, the smallest and largest quadrupole deformation parameter β_2 are 0.1580 ± 0.0032 and 0.2443 ± 0.0030 for ^{96}Ru and ^{102}Ru, respectively. Further, the pairing of like nucleons plays an important role in all $\beta\beta$ decay emitters, which are even-Z and even-N nuclei. Thus, it is expected that pairing and deformation degrees of freedom will play some crucial role in the structure of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei. For the study of 2ν e^+DBD modes of these nuclei, it is desirable to have a framework in which the pairing and deformation degrees of freedom are treated on equal footing in its formalism. The effects of deformation on the distribution of the Gamow-Teller (GT) and beta decay properties have been studied using a quasi-particle Tamm-Dancoff approximation (TDA) based on deformed Hartree-Fock (HF) calculations with Skyrme interactions [33], a deformed selfconsistent HF+RPA method with Skyrme-type interactions [34]. The comparison of the experimental GT strength distribution $B(GT)$ from its decay with the results of QRPA calculations was employed as a novel method of deducing the deformation of the $N = Z$ nucleus ^{76}Sr [35]. The effect of deformation on the 2ν $\beta\beta$ decay for ground-state transition $^{76}\mathrm{Ge} \to {}^{76}\mathrm{Se}$ was studied in the framework of the deformed QRPA with separable GT residual interaction [36]. A deformed QRPA formalism to describe simultaneously the energy distributions of the single-beta GT strength and the 2ν $\beta\beta$ decay matrix elements, using deformed Woods-Saxon potentials and deformed Skyrme Hartree-Fock mean fields was developed [37]. In all these works calculations are performed in the intrinsic basis, where angular momentum is not a good quantum number. The projected Hartree-Fock-Bogoliubov (PHFB) model offers, in this sense, a sensible alternative. However, in the present version of the PHFB model, it is not possible to study the structure of odd-odd nuclei. Hence, the single-beta decay rates and the distribution of Gamow-Teller strength cannot be calculated. On the other hand, the study of these processes has implications in the understanding of the role of the isoscalar part of the proton-neutron interaction. This is a serious draw back in the present formalism of the PHFB model. Notwithstanding, the PHFB model has been successfully applied to the $\beta^-\beta^-$ decay of many emitters in this mass region, where it was possible to describe, in the same context, the lowest excited states of the parent and daughter nuclei, as well as their electromagnetic transition strengths on one hand, and to reproduce their measured $\beta\beta$ decay rates on the other [38].

The aim of nuclear many-body theory is to describe the observed properties of nuclei in a coherent framework. The $\beta\beta$ decay can be studied in the same framework as many other nuclear properties and decays. Experimental studies involving in-beam γ-ray spectroscopy concerning the level energies as well as electromagnetic properties have yielded a vast amount of data over the past years. Although the availability of data permits a rigorous and detailed critique of the ingredients of the microscopic model that seeks to provide a description of nuclear $\beta\beta$ decay, most of the calculations of 2ν e^+DBD transition matrix elements performed so far but for the work of Barabash *et al.* [26] and Suhonen *et al.* [39] do not satisfy this criterion. The successful study of 2ν e^+DBD modes of ^{106}Cd for the $0^+ \to 0^+$ transition together with other observed nuclear properties, like the yrast spectra, reduced transition probabilities $B(E2{:}0^+ \to 2^+)$, static quadrupole moments $Q(2^+)$ and g-factors $g(2^+)$ of both parent and daughter nuclei using the PHFB model in conjunction with the summation method [40], has motivated us to apply the same framework to study the 2ν e^+DBD modes of ^{96}Ru, ^{102}Pd and ^{108}Cd isotopes. The reason for presenting again the results of ^{106}Cd is that the HFB wave functions are generated with improved accuracy and it is nice to see that the results remain almost unchanged.

Further, it has been shown that there exists an inverse correlation between the GT strength and the quadrupole moment [41,42]. The $PPQQ$ interaction [43] has two terms, associated with the pairing interaction (PP) and the quadrupole-quadrupole (QQ) interactions. The former accounts for the sphericity of nucleus, whereas the latter increases the collectivity in the nuclear intrinsic wave functions and makes the nucleus deformed. Hence, the PHFB model in conjunction with the $PPQQ$ interaction is a convenient choice to examine the explicit role of deformation on the NTME $M_{2\nu}$. In case of ^{106}Cd, we have already shown that deformation plays an important role in the variation of $M_{2\nu}$ *vis-à-vis* changing strength of the QQ part of effective two-body interaction [40].

The structure of the present paper is as follows. The theoretical formalism to calculate the half-lives of 2ν e^+DBD modes has been given in a number of reviews [4, 7] and in our earlier study of 2ν e^+DBD modes of ^{106}Cd for the $0^+ \to 0^+$ transition [40]. Hence, we briefly outline steps of the above derivations in sect. 2 for clarity of notation. Details of the mathematical expressions used to calculate the spectroscopic properties of nuclei in the PHFB model have been given by Dixit *et al.* [44]. In sect. 3, we present results to check the reliability of the wave functions of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei by calculating the mentioned spectroscopic properties and by comparing them with the available experimental data. The half-lives $T_{1/2}^{2\nu}$ for the 2ν e^+DBD modes of ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd nuclei for the $0^+ \to 0^+$ transition are calculated. The role of deformation on $M_{2\nu}$ is also studied. We present some concluding remarks in sect. 4.

2 Theoretical framework

The inverse half-life of the 2ν e^+DBD mode for the $0^+ \to 0^+$ transition is given by

$$\left[T_{1/2}^{2\nu}(0^+ \to 0^+)\right]^{-1} = G_{2\nu}\left|M_{2\nu}\right|^2, \qquad (1)$$

where $G_{2\nu}$ is the integrated kinematical factor and the NTME $M_{2\nu}$ is expressed as

$$M_{2\nu} = \sum_N \frac{\langle 0_F^+||\boldsymbol{\sigma}\tau^-||1_N^+\rangle\langle 1_N^+||\boldsymbol{\sigma}\tau^-||0_I^+\rangle}{E_0 + E_N - E_I}, \qquad (2)$$

where

$$E_0 = \frac{1}{2}\left(E_I - E_F\right) = \frac{1}{2}W_0. \qquad (3)$$

The total energy released, W_0, for different 2ν e^+DBD modes is given by

$$W_0(\beta^+\beta^+) = Q_{\beta^+\beta^+} + 2m_e, \qquad (4)$$

$$W_0(\beta^+EC) = Q_{\beta^+EC} + e_b, \qquad (5)$$

$$W_0(ECEC) = Q_{ECEC} - 2m_e + e_{b1} + e_{b2}. \qquad (6)$$

The summation over intermediate states is carried out using the summation method [45] and the NTME $M_{2\nu}$ can be written as

$$M_{2\nu} = \frac{1}{E_0}\left\langle 0_F^+ \left| \sum_m (-1)^m \Gamma_{-m} F_m \right| 0_I^+ \right\rangle, \qquad (7)$$

where the Gamow-Teller (GT) operator Γ_m has been defined as

$$\Gamma_m = \sum_s \boldsymbol{\sigma}_{ms}\tau_s^-, \qquad (8)$$

and

$$F_m = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{E_0^\lambda} D_\lambda \Gamma_m, \qquad (9)$$

with

$$D_\lambda \Gamma_m = [H,[H,........,\ [H,\Gamma_m]\,.......]]^{(\lambda \text{ times})}. \qquad (10)$$

When the GT operator commutes with the effective two-body interaction, eq. (7) can be further simplified to

$$M_{2\nu} = \sum_{\pi,\nu} \frac{\langle 0_F^+||\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\tau^-\tau^-||0_I^+\rangle}{E_0 + \varepsilon(n_\nu,l_\nu,j_\nu) - \varepsilon(n_\pi,l_\pi,j_\pi)}. \qquad (11)$$

The energy denominator is evaluated as follows. The difference in single-particle energies of neutrons in the intermediate nucleus and protons in the parent nucleus is mainly due to the difference in Coulomb energies. Hence

$$\varepsilon(n_\nu,l_\nu,j_\nu) - \varepsilon(n_\pi,l_\pi,j_\pi) = \begin{cases} \Delta_C - 2E_0 & \text{for } n_\nu = n_\pi,\ l_\nu = l_\pi, j_\nu = j_\pi \\ \Delta_C - 2E_0 + \Delta E_{s.o.splitting} & \text{for } n_\nu = n_\pi,\ l_\nu = l_\pi, j_\nu \neq j_\pi \end{cases}, \qquad (12)$$

where the Coulomb energy difference Δ_C is given by Bohr and Mottelson [46]

$$\Delta_C = \frac{0.70}{A^{1/3}}\left[(2Z+1) - 0.76\left\{(Z+1)^{4/3} - Z^{4/3}\right\}\right]. \qquad (13)$$

In the case of the pseudo $SU(3)$ model [47–49], the energy denominator is a well-defined quantity without any free parameter as the GT operator commutes with the two-body interaction. The energy denominator was evaluated exactly for 2ν $\beta^-\beta^-$ [47,48] and 2ν e^+DBD modes [49] in the pseudo $SU(3)$ scheme. It must be underlined that, in the present context, the use of the summation method goes beyond the closure approximation, because each proton-neutron excitation is weighted depending on its spin-flip or non–spin-flip character. The explicit inclusion of the spin-orbit splitting in the energy denominator, eq. (12), implies that it cannot be factorized out of the sum in eq. (11). In this sense, employing the summation method in conjunction with the PHFB formalism is richer than what was done in the previous application with the pseudo $SU(3)$ model [47,48].

In the present work, we use a Hamiltonian with $PPQQ$ type [43] of effective two-body interaction. Explicitly, the Hamiltonian is written as

$$H = H_{sp} + V(P) + \chi_{qq}V(QQ), \qquad (14)$$

where H_{sp} denotes the single-particle Hamiltonian. The pairing part of the effective two-body interaction $V(P)$ is written as

$$V(P) = -\left(\frac{G}{4}\right)\sum_{\alpha\beta}(-1)^{j_\alpha+j_\beta-m_\alpha-m_\beta} a_\alpha^\dagger a_{\bar{\alpha}}^\dagger a_{\bar{\beta}} a_\beta, \qquad (15)$$

where α denotes the quantum numbers $(nljm)$. The state $\bar{\alpha}$ is same as α but with the sign of m reversed. The QQ part of the effective interaction $V(QQ)$ is expressed as

$$V(QQ) = -\left(\frac{\chi}{2}\right)\sum_{\alpha\beta\gamma\delta}\sum_{\mu}(-1)^{\mu}\langle\alpha|q_{\mu}^{2}|\gamma\rangle\langle\beta|q_{-\mu}^{2}|\delta\rangle\; a_{\alpha}^{\dagger}a_{\beta}^{\dagger}\; a_{\delta}\; a_{\gamma}\,, \quad (16)$$

where

$$q_{\mu}^{2} = \left(\frac{16\pi}{5}\right)^{1/2} r^{2}Y_{\mu}^{2}(\theta,\phi). \quad (17)$$

The χ_{qq} is an arbitrary adimensional parameter and the final results are obtained by setting the $\chi_{qq} = 1$. The purpose of introducing χ_{qq} is to study the role of deformation by varying the strength of QQ part of the effective two-body interaction.

The model Hamiltonian given by eq. (14) is not isospin symmetric. Hence, the energy denominator is not as simple as in eq. (11). However, the violation of isospin symmetry for the QQ part of our model Hamiltonian is negligible, as will be evident from the parameters of the two-body interaction given later. Further, the violation of isospin symmetry for the pairing part of the two-body interaction is presumably small in the mass region under study. Under these assumptions, the expression to calculate the NTME $M_{2\nu}$ of 2ν e^{+}DBD modes for the $0^{+} \rightarrow 0^{+}$ transition in the PHFB model is obtained as follows.

In the PHFB model, states with good angular momentum $\mathbf{J}$ are obtained from the axially symmetric HFB intrinsic state $|\Phi_{0}\rangle$ with $K = 0$ using the standard projection technique [50] given by

$$|\Psi_{00}^{J}\rangle = \left[\frac{(2J+1)}{8\pi^{2}}\right]\int D_{00}^{J}(\Omega)R(\Omega)|\Phi_{0}\rangle\mathrm{d}\Omega, \quad (18)$$

where $R(\Omega)$ and $D_{00}^{J}(\Omega)$ are the rotation operator and the rotation matrix, respectively. The axially symmetric HFB intrinsic state $|\Phi_{0}\rangle$ can be written as

$$|\Phi_{0}\rangle = \prod_{im}(u_{im} + v_{im}b_{im}^{\dagger}b_{i\bar{m}}^{\dagger})|0\rangle, \quad (19)$$

where the creation operators $b_{im}^{\dagger}$ and $b_{i\bar{m}}^{\dagger}$ are defined as

$$b_{im}^{\dagger} = \sum_{\alpha} C_{i\alpha,m}a_{\alpha m}^{\dagger}$$
$$\text{and}\quad b_{i\bar{m}}^{\dagger} = \sum_{\alpha}(-1)^{l+j-m}C_{i\alpha,m}a_{\alpha,-m}^{\dagger}. \quad (20)$$

The results of HFB calculations are summarized by the amplitudes (u_{im}, v_{im}) and expansion coefficients $C_{ij,m}$.

Finally, one obtains the following expression for the NTME $M_{2\nu}$ of the 2ν e^{+}DBD mode:

$$\begin{aligned}M_{2\nu} &= \sum_{\pi,\nu}\frac{\langle\Psi_{00}^{J_f=0}||\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\tau^{-}\tau^{-}||\Psi_{00}^{J_i=0}\rangle}{E_{0}+\varepsilon(n_{\nu},l_{\nu},j_{\nu})-\varepsilon(n_{\pi},l_{\pi},j_{\pi})}\\ &= \left[n_{Z-2,N+2}^{J_f=0}n_{Z,N}^{J_i=0}\right]^{-1/2}\int_{0}^{\pi} n_{(Z,N),(Z-2,N+2)}(\theta)\\ &\times\sum_{\alpha\beta\gamma\delta}\frac{\langle\alpha\beta\,|\boldsymbol{\sigma}_{1}\cdot\boldsymbol{\sigma}_{2}\tau^{-}\tau^{-}|\,\gamma\delta\rangle}{E_{0}+\varepsilon_{\alpha}(n_{\nu},l_{\nu},j_{\nu})-\varepsilon_{\gamma}(n_{\pi},l_{\pi},j_{\pi})}\\ &\times\sum_{\varepsilon\eta}\frac{\left(f_{Z-2,N+2}^{(\nu)*}\right)_{\varepsilon\beta}}{\left[1+F_{Z,N}^{(\nu)}(\theta)f_{Z-2,N+2}^{(\nu)*}\right]_{\varepsilon\alpha}}\\ &\times\frac{\left(F_{Z,N}^{(\pi)*}\right)_{\eta\delta}}{\left[1+F_{Z,N}^{(\pi)}(\theta)f_{Z-2,N+2}^{(\pi)*}\right]_{\gamma\eta}}\sin\theta\mathrm{d}\theta,\end{aligned} \quad (21)$$

where

$$\begin{aligned}n^{J} &= \int_{0}^{\pi}\left\{\det\left[1+F^{(\pi)}(\theta)f^{(\pi)\dagger}\right]\right\}^{1/2}\\ &\times\left\{\det\left[1+F^{(\nu)}(\theta)f^{(\nu)\dagger}\right]\right\}^{1/2}d_{00}^{J}(\theta)\sin(\theta)\mathrm{d}\theta,\end{aligned} \quad (22)$$

and

$$\begin{aligned}n_{(Z,N),(Z-2,N+2)}(\theta) &= \{\det[1+F_{Z,N}^{(\pi)}(\theta)f_{Z-2,N+2}^{(\pi)\dagger}]\}^{1/2}\\ &\times\{\det[1+F_{Z,N}^{(\nu)}(\theta)f_{Z-2,N+2}^{(\nu)\dagger}]\}^{1/2}.\end{aligned} \quad (23)$$

Here $\pi(\nu)$ represents the proton (neutron) of nuclei involved in the 2ν e^{+}DBD. The matrices $f_{Z,N}$ and $F_{Z,N}(\theta)$ are given by

$$[f_{Z,N}]_{\alpha\beta} = \sum_{i} C_{ij_{\alpha},m_{\alpha}}C_{ij_{\beta},m_{\beta}}\left(v_{im_{\alpha}}/u_{im_{\alpha}}\right)\delta_{m_{\alpha},-m_{\beta}}, \quad (24)$$

and

$$[F_{Z,N}(\theta)]_{\alpha\beta} = \sum_{m'_{\alpha}m'_{\beta}} d_{m_{\alpha},m'_{\alpha}}^{j_{\alpha}}(\theta)d_{m_{\beta},m'_{\beta}}^{j_{\beta}}(\theta)f_{j_{\alpha}m'_{\alpha},j_{\beta}m'_{\beta}}. \quad (25)$$

The calculation of the NTME $M_{2\nu}$ for the 2ν e^{+}DBD mode is carried on as follows. In the first step, the matrices $[f_{Z,N}]_{\alpha\beta}$ and $[F_{Z,N}(\theta)]_{\alpha\beta}$ are set up using expressions given by eqs. (24) and (25), respectively. Finally, the required NTME $M_{2\nu}$ is calculated in a straightforward manner using eq. (21) with 20 Gaussian quadrature points in the range $(0, \pi)$.

3 Results and discussions

The model space, single-particle energies (SPEs) and the effective two-body interaction are the same employed in our earlier calculation on 2ν e^{+}DBD modes of ^{106}Cd for the $0^+ \to 0^+$ transition [40]. However, we present a brief discussion of them in the following for convenience. The model space consists of $1p_{1/2}$, $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$, $0g_{9/2}$ and $0h_{11/2}$ orbits for protons and neutrons, where we have treated the doubly even nucleus ^{76}Sr ($N = Z = 38$) as an inert core. The orbit $1p_{1/2}$ has been included in the valence space to examine the role of the $Z = 40$ proton core *vis-à-vis* the onset of deformation in highly neutron-rich isotopes. The set of single-particle energies (SPEs) used here are (in MeV) $\varepsilon(1p_{1/2}) = -0.8$, $\varepsilon(0g_{9/2}) = 0.0$, $\varepsilon(1d_{5/2}) = 5.4$, $\varepsilon(2s_{1/2}) = 6.4$, $\varepsilon(1d_{3/2}) = 7.9$, $\varepsilon(0g_{7/2}) = 8.4$ and $\varepsilon(0h_{11/2}) = 8.6$ for proton and neutrons. This set of SPEs but for $\varepsilon(0h_{11/2})$, which is slightly lowered, has been employed in a number of successful shell model [51] as well as variational model [52] calculations for nuclear properties in the mass region $A \approx 100$.

The strengths of the pairing interaction has been fixed through the relations $G_p = 30/A$ MeV and $G_n = 20/A$ MeV, which are the same as those used by Heestand *et al.* [53] to explain the experimental $g(2^+)$ data of some even-even Ge, Se, Mo, Ru, Pd, Cd and Te isotopes in Greiner's collective model [54]. The strengths of the like particle components of the QQ interaction are taken as $\chi_{pp} = \chi_{nn} = 0.0105\,\text{MeV}\ b^{-4}$, where b is the oscillator parameter. The strength of the proton-neutron (pn) component of the QQ interaction χ_{pn} is varied to fit the spectra of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd in agreement with the experimental results. To be more specific, we have taken the theoretical spectra to be the optimum ones if the excitation energy of the 2^+ state E_{2^+} is reproduced as closely as possible to the experimental value. Thus for a given model space, SPEs, G_p, G_n and χ_{pp}, we have fixed χ_{pn} through the experimentally available energy spectra. We have given the values of χ_{pn} in table 1. These values for the strength of the QQ interaction are comparable to those suggested by Arima on the basis of an empirical analysis of the effective two-body interactions [55]. All the parameters are kept fixed throughout the calculation.

3.1 The yrast spectra and electromagnetic properties

In table 1 we have displayed the theoretically calculated and experimentally observed values of yrast spectra for $J^\pi = 2^+, 4^+$ and 6^+ states of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd isotopes. The agreement between the experimentally observed [56] and theoretically reproduced E_{2^+} is quite good. However, it can be noticed that the theoretical spectra are more expanded in comparison with the experimental spectra. This can be corrected to some extent in the PHFB model in conjunction with the VAP prescription [52]. However, our aim is to reproduce properties of the low-lying 2^+ state. Hence, we have not attempted to invoke the VAP prescription, which will unnecessarily complicate the calculations.

In table 2 we present the calculated as well as the experimentally observed values of the reduced $B(E2{:}0^+ \to 2^+)$ transition probabilities [57], static quadrupole moments $Q(2^+)$ and the gyromagnetic factors $g(2^+)$ [58]. In case of $B(E2{:}0^+ \to 2^+)$, only some representative experimental values are tabulated. $B(E2{:}0^+ \to 2^+)$ results are given for effective charges $e_{eff} = 0.40$, 0.50 and 0.60 in columns 2 to 4, respectively. The experimentally observed values are displayed in column 5. The calculated values are in excellent agreement with the observed $B(E2{:}0^+ \to 2^+)$ of all the nuclei considered at $e_{eff} = 0.50$ except for ^{102}Ru, ^{102}Pd and ^{108}Pd, which differ by 0.049, 0.02 and 0.046 e^2 b^2, respectively, from the experimental lower limits.

Table 1. Excitation energies (in MeV) of $J^\pi = 2^+$, 4^+ and 6^+ yrast states of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei.

Nucleus	χ_{pn}		Theo.	Expt. [56]
$^{96}_{44}$Ru	0.02417	E_{2^+}	0.8323	0.8326
		E_{4^+}	2.1389	1.51797
		E_{6^+}	3.8037	2.1496
$^{102}_{46}$Pd	0.01573	E_{2^+}	0.5551	0.5565
		E_{4^+}	1.6010	1.2760
		E_{6^+}	2.9467	2.1115
$^{106}_{48}$Cd	0.01505	E_{2^+}	0.6321	0.6327
		E_{4^+}	1.7298	1.4939
		E_{6^+}	3.1610	2.4918
$^{108}_{48}$Cd	0.01481	E_{2^+}	0.6319	0.6330
		E_{4^+}	1.8072	1.5084
		E_{6^+}	3.3138	2.5413
$^{96}_{42}$Mo	0.02557	E_{2^+}	0.7779	0.7782
		E_{4^+}	2.0373	1.6282
		E_{6^+}	3.5775	2.4406
$^{102}_{44}$Ru	0.02054	E_{2^+}	0.4751	0.4751
		E_{4^+}	1.4773	1.1064
		E_{6^+}	2.8737	1.8732
$^{106}_{46}$Pd	0.01441	E_{2^+}	0.5115	0.5119
		E_{4^+}	1.4816	1.2292
		E_{6^+}	2.7264	2.0766
$^{108}_{46}$Pd	0.01443	E_{2^+}	0.4336	0.4339
		E_{4^+}	1.3126	1.0482
		E_{6^+}	2.4826	1.7712

The theoretically calculated $Q(2^+)$ values are tabulated in columns 6 to 8 of the same table 2, along with the experimentally observed $Q(2^+)$ data in column 9, for the same effective charges as those used in case of $B(E2{:}0^+ \to 2^+)$. Again, the agreement between the calculated and experimental $Q(2^+)$ values is quite good in case of ^{102}Ru, ^{108}Cd, ^{106}Pd and ^{108}Pd nuclei except for ^{106}Cd, where the difference is 0.2 e b. In case of ^{96}Ru,^{96}Mo and ^{102}Pd, although the experimental values have large error bars and a meaningful comparison is difficult, the agreement between calculated and observed values is not satisfactory. The $g(2^+)$ values are calculated with $g_l^\pi = 1.0$, $g_l^\nu = 0.0$, $g_s^\pi = g_s^\nu = 0.60$. No experimental result for $g(2^+)$ is available for the isotope ^{96}Ru. The calculated

Table 2. Comparison of the calculated and experimentally observed reduced transition probability $B(E2{:}0^+ \to 2^+)$ in e^2 b^2, static quadrupole moments $Q(2^+)$ in e b and g-factors $g(2^+)$ in nuclear magneton. Here $B(E2)$ and $Q(2^+)$ are calculated for the effective charges $e_p = 1 + e_{eff}$ and $e_n = e_{eff}$. The $g(2^+)$ has been calculated for $g_l^\pi = 1.0$, $g_l^\nu = 0.0$ and $g_s^\pi = g_s^\nu = 0.60$.

Nuclei	$B(E2{:}0^+ \to 2^+)$				$Q(2^+)$				$g(2^+)$	
	Theo.			Expt. [57]	Theo.			Expt. [58]	Theo.	Expt. [58]
	e_{eff}				e_{eff}					
	0.40	0.50	0.60		0.40	0.50	0.60			
^{96}Ru	0.208	0.261	0.319	0.251 ± 0.010	−0.412	−0.461	−0.510	-0.15 ± 0.27	0.394	
				0.260 ± 0.010						
^{96}Mo	0.265	0.335	0.413	0.310 ± 0.047	−0.466	−0.524	−0.582	-0.20 ± 0.08	0.563	$0.419 \pm 0.033 \pm 0.038^*$
				0.271 ± 0.005				$+0.04 \pm 0.08$		
^{102}Pd	0.323	0.410	0.507	0.460 ± 0.030	−0.514	−0.580	−0.645	-0.20 ± 0.20	0.386	0.41 ± 0.04
										0.39 ± 0.05
^{102}Ru	0.458	0.585	0.726	0.640 ± 0.006	−0.613	−0.692	−0.771	-0.57 ± 0.07	0.385	0.371 ± 0.031
				0.651 ± 0.016				-0.68 ± 0.08		
^{106}Cd	0.330	0.422	0.525	0.410 ± 0.020	−0.518	−0.586	−0.654	-0.28 ± 0.08	0.372	0.40 ± 0.10
				0.386 ± 0.05						
^{106}Pd	0.403	0.515	0.640	0.610 ± 0.090	−0.573	−0.648	−0.722	-0.56 ± 0.08	0.465	0.398 ± 0.021
				0.656 ± 0.035				-0.51 ± 0.08		0.30 ± 0.06
^{108}Cd	0.414	0.531	0.661	0.540 ± 0.011	−0.581	−0.657	−0.734	-0.45 ± 0.08	0.361	0.34 ± 0.09
				0.430 ± 0.020						
^{108}Pd	0.456	0.584	0.727	0.700 ± 0.070	−0.610	−0.690	−0.770	-0.58 ± 0.04	0.483	0.36 ± 0.03
				0.760 ± 0.040				-0.51 ± 0.06		0.32 ± 0.03

*P.F. Mantica *et al.*, Phys. Rev. C **63**, 034312 (2001).

and experimentally observed $g(2^+)$ values are in excellent agreement for ^{102}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd nuclei whereas they are off by 0.073, 0.046 and 0.093 nm only for ^{96}Mo, ^{106}Pd and ^{108}Pd isotopes, respectively.

From the above discussions, it is clear that the overall agreement between the calculated and observed electromagnetic properties is quite good. Hence, the PHFB wave functions of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei generated by fixing χ_{pn} to reproduce the yrast spectra are quite reliable. Below, we present the results of NTMEs $M_{2\nu}$ as well as the half-lives $T_{1/2}^{2\nu}$ of ^{96}Ru, ^{102}Pd,^{106}Cd and ^{108}Cd for the $0^+ \to 0^+$ transition using the same HFB wave functions.

3.2 Results of 2ν $\beta^+\beta^+/\beta^+$EC/ECEC decay

In table 3, we have compiled the available experimental and theoretical results for 2ν e$^+$DBD modes of ^{96}Ru, ^{102}Pd,^{106}Cd and ^{108}Cd nuclei along with our calculated NTMEs $M_{2\nu}$ and the corresponding half-lives $T_{1/2}^{2\nu}$. The calculated phase space factors were obtained following the prescription of Doi *et al.* [4] in the approximation $C_1 = 1.0$, $C_2 = 0.0$, $C_3 = 0.0$ and $R_{1,1}(\varepsilon) = R_{+1}(\varepsilon) + R_{-1}(\varepsilon) = 1.0$. The phase space integrals have been evaluated for $g_A = 1.261$ by Doi *et al.* [4]. However, in heavy nuclei it is more justified to use the nuclear matter value of g_A around 1.0. Hence, the theoretical $T_{1/2}^{2\nu}$ are presented both for $g_A = 1.0$ and 1.261.

In the case of ^{96}Ru, the half-life limits $T_{1/2}^{2\nu}$ of the 2ν β^+EC and 2ν $ECEC$ modes for the $0^+ \to 0^+$ transition have been investigated by Norman [21] and are of the order of 10^{16} y. The calculated NTMEs $M_{2\nu}$ in the PHFB and $SU(4)_{\sigma\tau}$ [59] models differ by a factor of 2 for all the three modes, while in QRPA model [60], the values of NTMEs $M_{2\nu}$ are larger than the PHFB model values by a factor of 5, approximately. The phase space factors for the ^{96}Ru isotope are $G_{2\nu}(\beta^+\beta^+) = 2.516 \times 10^{-26}\,\mathrm{y}^{-1}$, $G_{2\nu}(\beta^+EC) = 9.635 \times 10^{-22}\,\mathrm{y}^{-1}$ and $G_{2\nu}(ECEC) = 5.385 \times 10^{-21}\,\mathrm{y}^{-1}$. The theoretically calculated $T_{1/2}^{2\nu}$ are of the order of 10^{26-28} y, 10^{22-23} y and 10^{21-23} y for 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν $ECEC$ modes, respectively, for $g_A = 1.261$–1.00.

The e$^+$DBD modes of the ^{102}Pd isotope for the $0^+ \to 0^+$ transition have been investigated neither experimentally nor theoretically so far. We have used the phase space factors $G_{2\nu}(\beta^+EC) = 1.449 \times 10^{-30}\,\mathrm{y}^{-1}$ and $G_{2\nu}(ECEC) = 9.611 \times 10^{-23}\,\mathrm{y}^{-1}$ for the 2ν β^+EC and 2ν $ECEC$ modes, respectively. In the PHFB model, the predicted $T_{1/2}^{2\nu}$ of the 2ν β^+EC and 2ν $ECEC$ modes are $(2.509$–$6.344) \times 10^{32}$ y and $(3.783$–$9.565) \times 10^{24}$ y, respectively, for $g_A = 1.261$–1.00.

We have compiled the available experimental [12,20, 24–30] and theoretical results [26,39,59–65] for ^{106}Cd

Table 3. Experimental limits on half-lives $T_{1/2}^{2\nu}(0^+ \to 0^+)$, theoretically calculated $M_{2\nu}$ and corresponding $T_{1/2}^{2\nu}(0^+ \to 0^+)$ for 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν $ECEC$ decay of ^{96}Ru, ^{102}Pd, ^{106}Cd and ^{108}Cd nuclei. Half-lives are calculated using $g_A = (1.261–1.0)$, respectively. * and ** denote the half-life limit for $0\nu + 2\nu$ and $0\nu + 2\nu + 0\nu$M modes, respectively.

Nuclei	Decay mode	Experiment		Theory			
		Ref.	$T_{1/2}^{2\nu}$ (y)	Ref.	Model	$\|M_{2\nu}\|$	$T_{1/2}^{2\nu}$ (y)
^{96}Ru	$\beta^+\beta^+$	[21]	$> 3.1 \times 10^{16*}$	Present	PHFB	0.0537	$(1.378–3.485) \times 10^{28}$
				[60]	QRPA	0.2510	$(6.309–15.950) \times 10^{26}$
	β^+EC	[21]	$> 6.7 \times 10^{16*}$	Present	PHFB	0.0537	$(3.599–9.100) \times 10^{23}$
				[59]	$SU(4)_{\sigma\tau}$	0.1005	$(1.028–2.598) \times 10^{23}$
				[60]	QRPA	0.2694	$(1.430–3.616) \times 10^{22}$
	$ECEC$		–	Present	PHFB	0.0537	$(0.644–1.628) \times 10^{23}$
				[59]	$SU(4)_{\sigma\tau}$	0.1005	$(1.839–4.649) \times 10^{22}$
				[60]	QRPA	0.2620	$(2.705–6.840) \times 10^{21}$
^{102}Pd	β^+EC			Present	PHFB	0.0524	$(2.509–6.344) \times 10^{32}$
	$ECEC$			Present	PHFB	0.0524	$(3.783–9.565) \times 10^{24}$
^{106}Cd	$\beta^+\beta^+$	[29]	$> 5.0 \times 10^{18}$	Present	PHFB	0.0819	$(3.495–8.836) \times 10^{27}$
		[27]	$> 2.4 \times 10^{20**}$	[65]	SQRPA(l.b.)	0.61	$(6.304–15.940) \times 10^{25}$
		[26]	$> 1.0 \times 10^{19*}$		SQRPA(s.b.)	0.57	$(7.220–18.260) \times 10^{25}$
		[25]	$> 9.2 \times 10^{17}$	[39]	QRPA(AWS)	0.722	$(4.500–11.380) \times 10^{25}$
		[20]	$> 2.6 \times 10^{17*}$		QRPA(WS)	0.166	$(8.513–21.520) \times 10^{26}$
		[12]	$> 6 \times 10^{16}$	[26]	QRPA(WS)	0.840	$(3.324–8.406) \times 10^{25}$
					QRPA(AWS)	0.780	$(3.856–9.749) \times 10^{25}$
				[60]	QRPA	0.218	$(4.936–12.480) \times 10^{26}$
				[61]	QRPA		4.940×10^{25}
	β^+EC	[29]	$> 1.2 \times 10^{18}$	Present	PHFB	0.0819	$(9.489–23.992) \times 10^{22}$
		[27]	$> 4.1 \times 10^{20}$	[65]	SQRPA(l.b.)	0.61	$(1.712–4.328) \times 10^{21}$
		[26]	$> 0.66 \times 10^{19*}$		SQRPA(s.b.)	0.57	$(1.960–4.957) \times 10^{21}$
		[25]	$> 2.6 \times 10^{17}$	[39]	QRPA(AWS)	0.718	$(1.236–3.124) \times 10^{21}$
		[20]	$> 5.7 \times 10^{17*}$		QRPA(WS)	0.168	$(2.257–5.706) \times 10^{22}$
				[59]	$SU(4)_{\sigma\tau}$	0.1947	$(1.680–4.248) \times 10^{22}$
				[63]	RQRPA(AWS)	0.56	$(2.031–5.136) \times 10^{21}$
					RQRPA(WS)	0.55	$(2.106–5.324) \times 10^{21}$
				[26]	QRPA(WS)	0.84	$(9.027–22.820) \times 10^{20}$
					QRPA(AWS)	0.78	$(1.047–2.647) \times 10^{21}$
				[60]	QRPA	0.352	$(5.141–13.000) \times 10^{21}$
				[62]	QRPA(WS)	0.493	$(2.621–6.626) \times 10^{21}$
						0.660	$(1.462–3.697) \times 10^{21}$
^{106}Cd	$ECEC$	[30]	$> 1.0 \times 10^{18}$	Present	PHFB	0.0819	$(1.293–3.270) \times 10^{22}$
		[29]	$> 5.8 \times 10^{17}$	[65]	SQRPA(l.b.)	0.61	$(2.333–5.899) \times 10^{20}$
		[28]	$> 1.0 \times 10^{18}$		SQRPA(s.b.)	0.57	$(2.672–6.756) \times 10^{20}$
		[24]	$> 5.8 \times 10^{17}$	[39]	QRPA(AWS)	0.718	$(1.684–4.258) \times 10^{20}$
					QRPA(WS)	0.168	$(3.076–7.780) \times 10^{21}$
				[64]	SSDH(Theo)	0.28	$(1.107–2.800) \times 10^{21}$
					SSDH(Exp)	0.17	$(3.004–7.595) \times 10^{21}$
				[59]	$SU(4)_{\sigma\tau}$	0.1947	$(2.290–5.790) \times 10^{21}$
				[63]	RQRPA(AWS)	0.56	$(2.768–6.999) \times 10^{20}$
					RQRPA(WS)	0.55	$(2.870–7.256) \times 10^{20}$
				[26]	QRPA(WS)	0.84	$(1.230–3.111) \times 10^{20}$
					QRPA(AWS)	0.78	$(1.427–3.608) \times 10^{20}$
				[60]	QRPA	0.270	$(1.191–3.011) \times 10^{21}$
				[62]	QRPA(WS)	0.493	$(3.572–9.031) \times 10^{20}$
						0.660	$(1.993–5.039) \times 10^{20}$
^{108}Cd	$ECEC$	[29]	$> 4.1 \times 10^{17}$	Present	PHFB	0.0952	$(3.939–9.959) \times 10^{27}$
		[28]	$> 1.0 \times 10^{18}$				
		[24]	$> 4.1 \times 10^{17}$				

Table 4. Effect of the variation in χ_{qq} on $\langle Q_0^2 \rangle$, β_2 and NTMEs $M_{2\nu}$.

	χ_{qq}	0.00	0.20	0.40	0.60	0.80	0.90	0.95	1.00	1.05	1.10	1.20	1.30	1.40	1.50
^{96}Ru	$\langle Q_0^2 \rangle$	0.0	0.006	0.214	0.144	23.854	30.351	32.485	34.473	36.42	38.15	66.54	70.05	73.61	77.48
	β_2	0.0	0.046	0.097	0.098	0.112	0.140	0.151	0.161	0.171	0.180	0.295	0.317	0.337	0.348
^{96}Mo	$\langle Q_0^2 \rangle$	0.0	0.695	0.211	0.477	22.464	31.82	37.02	41.73	45.15	48.15	61.43	65.44	66.70	67.64
	β_2	0.0	0.091	0.093	0.093	0.106	0.149	0.174	0.191	0.210	0.224	0.268	0.281	0.286	0.290
	$M_{2\nu}$	0.168	0.154	0.152	0.153	0.093	0.072	0.067	0.054	0.036	0.024	0.076	0.049	0.056	0.049
^{102}Pd	$\langle Q_0^2 \rangle$	0.0	0.252	0.081	1.080	1.839	36.08	42.32	45.47	47.91	49.63	52.67	56.37	84.71	85.71
	β_2	0.0	0.085	0.046	0.090	0.092	0.149	0.172	0.185	0.196	0.203	0.217	0.234	0.349	0.353
^{102}Ru	$\langle Q_0^2 \rangle$	0.0	0.028	0.123	0.492	38.04	49.81	53.57	56.51	58.87	61.19	66.93	88.17	88.73	89.25
	β_2	0.0	0.036	0.068	0.092	0.159	0.204	0.220	0.232	0.242	0.252	0.279	0.362	0.365	0.367
	$M_{2\nu}$	0.178	0.203	0.208	0.215	0.135	0.092	0.072	0.052	0.039	0.027	0.021	0.0001	0.015	0.014
^{106}Cd	$\langle Q_0^2 \rangle$	0.0	0.008	0.031	0.128	0.510	32.65	40.83	47.14	55.89	62.54	73.53	83.65	90.63	91.33
	β_2	0.0	0.007	0.003	0.035	0.073	0.127	0.152	0.176	0.211	0.243	0.299	0.325	0.344	0.347
^{106}Pd	$\langle Q_0^2 \rangle$	0.0	0.022	0.079	0.192	0.897	39.54	46.88	52.12	56.13	59.31	66.23	73.87	79.84	92.21
	β_2	0.0	0.016	0.042	0.064	0.094	0.158	0.183	0.203	0.216	0.227	0.254	0.290	0.322	0.364
	$M_{2\nu}$	0.169	0.164	0.162	0.166	0.170	0.127	0.095	0.082	0.084	0.066	0.041	0.027	0.004	0.001
^{108}Cd	$\langle Q_0^2 \rangle$	0.0	0.015	0.047	0.132	0.488	35.73	43.87	53.29	67.98	76.17	79.26	80.61	84.18	94.34
	β_2	0.0	0.001	0.013	0.039	0.076	0.134	0.161	0.195	0.258	0.299	0.311	0.316	0.326	0.352
^{108}Pd	$\langle Q_0^2 \rangle$	0.0	0.047	0.100	0.267	28.21	44.76	50.99	55.88	59.63	62.85	70.17	76.61	83.14	86.42
	β_2	0.0	0.046	0.051	0.076	0.125	0.174	0.196	0.213	0.225	0.236	0.267	0.299	0.334	0.347
	$M_{2\nu}$	0.208	0.199	0.203	0.204	0.183	0.120	0.093	0.095	0.084	0.052	0.029	0.020	0.011	0.001

along with our calculated $M_{2\nu}$ and corresponding half-life $T_{1/2}^{2\nu}$ in table 3. In the case of ^{106}Cd, the phase factors are $G_{2\nu}(\beta^+\beta^+) = 4.263 \times 10^{-26}\,\mathrm{y}^{-1}$, $G_{2\nu}(\beta^+ EC) = 1.570 \times 10^{-21}\,\mathrm{y}^{-1}$and $G_{2\nu}(ECEC) = 1.152 \times 10^{-20}\,\mathrm{y}^{-1}$, respectively. In comparison with the theoretically predicted $T_{1/2}^{2\nu}$, the present experimental limits for the $0^+ \to 0^+$ transition of ^{106}Cd are smaller by a factor of 10^{5-7} in the case of the 2ν $\beta^+\beta^+$ mode but they are quite close for 2ν $\beta^+ EC$ and 2ν $ECEC$ modes. The half-life $T_{1/2}^{2\nu}$ calculated in the PHFB model using the summation method differs from all the existing calculations. The presently calculated NTME $M_{2\nu}$ is smaller than the recently given results in the QRPA(WS) model of Suhonen and Civitarese [39] by a factor of 2 approximately for all the three modes. The theoretical $M_{2\nu}$ values of the PHFB model and $SU(4)_{\sigma\tau}$ [59] again differ by a factor of 2 approximately for the 2ν $\beta^+ EC$ and 2ν $ECEC$ modes. On the other hand, the $M_{2\nu}$ calculated in our PHFB model is smaller than the values of Hirsch *et al.* [60] by a factor of 3 approximately in the case of the 2ν $\beta^+\beta^+$ and 2ν $ECEC$ modes while for the 2ν $\beta^+ EC$ mode the results differ by a factor of 4 approximately. All the rest of the calculations predict NTMEs which are larger than our predicted $M_{2\nu}$ approximately by a factor of 7 [62,63] to 10 [26]. The predicted $T_{1/2}^{2\nu}$ of the 2ν $\beta^+\beta^+$, 2ν $\beta^+ EC$ and 2ν $ECEC$ modes in the PHFB model are $(3.495\text{–}8.836) \times 10^{27}$ y, $(9.489\text{–}23.992) \times 10^{22}$ y and $(1.293\text{–}3.270) \times 10^{22}$ y, respectively, for $g_A = 1.261$ and 1.0.

The 2ν $ECEC$ mode of ^{108}Cd for the $0^+ \to 0^+$ transition has been investigated by Georgadze *et al.* [24], Kiel *et al.* [28] and Danevich *et al.* [29]. No theoretical calculation has been done so far to study the above-mentioned mode of the ^{108}Cd isotope. The phase space factor of the 2ν $ECEC$ mode is $G_{2\nu}(ECEC) = 2.803 \times 10^{-26}\,\mathrm{y}^{-1}$. In the PHFB model, the calculated half-life $T_{1/2}^{2\nu}$ of the 2ν $ECEC$ decay mode is 3.939×10^{27} y and 9.959×10^{27} y for $g_A = 1.261$ and 1.0, respectively.

The quenching of the nuclear matrix elements seems to be closely related with the explicit inclusion of deformation effects, which are absent in the other models. We analyze in detail this point below.

3.3 Deformation effect

We have investigated the variation of $\langle Q_0^2 \rangle$, β_2 and $M_{2\nu}$ with respect to the change in strength of the QQ interaction χ_{qq} to understand the role of deformation on the NTME $M_{2\nu}$. Out of several possibilities, we have taken the quadrupole moment of the intrinsic state $\langle Q_0^2 \rangle$ (in arbitrary units) and the quadrupole deformation parameter

β_2 as a quantitative measure of the deformation. The quadrupole moment of the intrinsic states $\langle Q_0^2 \rangle$, the deformation parameter β_2 and the NTMEs $M_{2\nu}$ for different χ_{qq} are tabulated in table 4. The deformation parameter has been calculated with the same effective charge as used in the calculation of $B(E2{:}0^+ \to 2^+)$ transition probabilities.

It is noticed that $\langle Q_0^2 \rangle$ as well as β_2 increases in general as χ_{qq} is varied from 0 to 1.5 except for a few anomalies. The intrinsic quadrupole moments show fluctuations in the case of ^{96}Ru at $\chi_{qq} = 0.6$. In the case of ^{96}Mo, similar fluctuations are observed at χ_{qq} equal to 0.4 and 0.6. In the case of ^{102,106}Pd nuclei, the fluctuations occur at $\chi_{qq} = 0.4$. In all cases, it is found that the quadrupole deformation parameter β_2 follows the same behavior as the quadrupole moment of the intrinsic state $\langle Q_0^2 \rangle$ with respect to the change in χ_{qq}, except for the ^{106}Cd isotope. In this case, $\langle Q_0^2 \rangle$ increases but β_2 decreases at $\chi_{qq} = 0.4$. Further, there is an anticorrelation between the deformation parameter and the NTME $M_{2\nu}$, in general, but for a few exceptions.

To quantify the effect of deformation on $M_{2\nu}$, we define a quantity $D_{2\nu}$ as the ratio of $M_{2\nu}$ at zero deformation ($\chi_{qq} = 0$) and full deformation ($\chi_{qq} = 1$). The $D_{2\nu}$ is given by

$$D_{2\nu} = \frac{M_{2\nu}(\chi_{qq} = 0)}{M_{2\nu}(\chi_{qq} = 1)}. \qquad (26)$$

The values of $D_{2\nu}$ are 3.13, 3.40, 2.06 and 2.19 for ^{96}Ru,^{102}Pd, ^{106}Cd and ^{108}Cd nuclei, respectively. These values of $D_{2\nu}$ suggest that $M_{2\nu}$ is quenched by a factor of 2 to 3.5 approximately in the mass region $96 \le A \le 108$ due to deformation effects.

Given the schematic nature of the $PPQQ$ interaction employed in the present calculation, and the fact that many of the nuclei studied are in the transitional region and do not display a well-defined rotational spectrum, the quenching factors discussed above could be considered as a conservative estimate of the uncertainties in the predicted 2ν nuclear matrix elements. They qualify both the present results and those obtained with other models where deformation is not explicitly considered. The uncertainties associated with the 0ν processes would be expected to be far smaller than in the 2ν mode.

In $\beta\beta$ decay studies where deformation was included but no angular momentum projection was performed, nuclear deformation was found to be a mechanism of suppression of the 2ν $\beta\beta$ decay. In this case, the $\beta\beta$ decay matrix elements are found to have maximum values for about equal deformations of parent and daughter nuclei, and they decrease rapidly when differences in deformations increase [37]. This deformation effect is different from the one reported in this work. Further research is needed to relate these two approaches.

4 Conclusions

To summarize, we have tested the quality of PHFB wave functions by comparing the theoretically calculated results for yrast spectra, reduced $B(E2{:}0^+ \to 2^+)$ transition probabilities, static quadrupole moments $Q(2^+)$ and g-factors $g(2^+)$ of ^{96,102}Ru, ^{96}Mo, ^{102,106,108}Pd and ^{106,108}Cd nuclei participating in 2ν e$^+$DBD modes with the available experimental results. The same PHFB wave functions are employed to calculate NTMEs $M_{2\nu}$ and half-lives $T_{1/2}^{2\nu}$ of ^{96}Ru (2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν $ECEC$ modes), ^{102}Pd (2ν β^+EC and 2ν $ECEC$ modes), ^{106}Cd (2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν $ECEC$ modes) and ^{108}Cd (2ν $ECEC$ mode) nuclei. It is noticed that the proton-neutron part of the $PPQQ$ interaction, which is responsible for triggering deformation in the intrinsic ground state, plays an important role in the quenching of $M_{2\nu}$ by a factor of approximately 2 to 3.5 in the considered mass region $96 \le A \le 108$. In the case of ^{96}Ru and ^{106}Cd, we have presented and discussed the theoretical results of 2ν e$^+$DBD modes in the PHFB model along with other available nuclear models for the $0^+ \to 0^+$ transition. In the case of ^{102}Pd and ^{108}Cd, these are the first theoretical calculations and in view of growing interests in the study of 2ν e$^+$DBD modes, these predictions would be helpful in the planning of future experimental set-ups.

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References

1. S.P. Rosen, H. Primakoff, in *Alpha-beta-gamma ray spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1965) p. 1499.
2. J.D. Vergados, Nucl. Phys. B **218**, 109 (1983).
3. J.D. Vergados, Phys. Rep. **133**, 1 (1986).
4. M. Doi, T. Kotani, Prog. Theor. Phys. **87**, 1207 (1992).
5. M. Doi, T. Kotani, Prog. Theor. Phys. **89**, 139 (1993).
6. A.S. Barabash, *Proceedings of the International Workshop on Double Beta Decay and Related Topics, Trento, Italy, 1995* (World Scientific, Singapore, 1996) p. 502.
7. J. Suhonen, O. Civitarese, Phys. Rep. **300**, 123 (1998).
8. I.V. Kirpichnikov, Phys. At. Nucl. **63**, 1341 (2000).
9. H.V. Klapdor-Kleingrothaus, *Sixty Years of Double Beta Decay* (World Scientific, Singapore, 2001).
10. V.I. Tretyak, Y.G. Zdesenko, At. Data Nucl. Data Tables **61**, 43 (1995); **80**, 83 (2002).
11. A.S. Barabash, Phys. At. Nucl. **67**, 438 (2004).
12. R.G. Winter, Phys. Rev. **99**, 88 (1955).
13. C.W. Kim, K. Kubodera, Phys. Rev. D **27**, 2765 (1983).
14. J. Abad, A. Morales, R. Nunez-Lagos, A.F. Pacheco, An. Fis. A **80**, 15 (1984); J. Phys. (Paris) **45**, C3-147 (1984).
15. Ya.V. Zeldovich, M.Yu. Khlopov, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 148 (1981) [JETP Lett. **34**, 141 (1981)].
16. R.A. Eramzhyan, G.V. Micelmacher, M.E. Voloshin, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 530 (1982).
17. J. Bernabeu, A. De Rujula, C. Jarlskog, Nucl. Phys. B **223**, 15 (1983).
18. S.K. Balaev, A.A. Kuliev, D.I. Salamov, Izv. Akad. Nauk USSR, Ser. Fiz. **53**, 2136 (1989) (in Russian).
19. E. Bellotti, E. Fiorini, C. Liguori, A. Pullia, A. Sarracino, L. Zanotti, Lett. Nuovo Cimento **33**, 273 (1982).
20. E.B. Norman, A. DeFaccio, Phys. Lett. B **148**, 31 (1984).

21. E.B. Norman, Phys. Rev. C **31**, 1937 (1985).
22. P. Vogel, M.R. Zirnbauer, Phys. Rev. Lett. **57**, 3148 (1986).
23. A.S. Barabash, JETP Lett. **51**, 207 (1990).
24. A.Sh. Georgadze, F.A. Danevich, Yu.G. Zdesenko, V.V. Kobychev, B.N. Kropivyanskii, V.N. Kuts, A.S. Nikolaiko, V.I. Tretyak, Phys. At. Nucl. **58**, 1093 (1995).
25. F.A. Danevich *et al.*, Z. Phys. A **355**; **433** (1996).
26. A.S. Barabash, V.I. Umatov, R. Gurriaran, F. Hubert, Ph. Hubert, M. Aunola, J. Suhonen, Nucl. Phys. A **604**, 115 (1996).
27. P. Belli, R. Bernabei, A. Incicchitti, C. Arpesella, V.V. Kobychev, O.A. Ponkratenko, V.I. Tretyak, Yu.G. Zdesenko, Astropart. Phys. **10**, 115 (1999).
28. H. Kiel, D. Münstermann, K. Zuber, Nucl. Phys. A **23**, 499 (2003); arXiv: nucl-ex/0301007.
29. F.A. Danevich, A.Sh. Georgadze, V.V. Kobychev, B.N. Kropivyansky, A.S. Nikolaiko, O.A. Ponkratenko, V.I. Tretyak, S.Yu. Zdesenko, Yu.G. Zdesenko, Phys. Rev. C **68**, 035501 (2003).
30. K. Zuber, Eur. Phys. J. C **33**, 817 (2004).
31. Yutaka Ito, Makoto Minowa, Wataru Ootani, Keiji Nishigaki, Yasuhiro Kishimoto, Takayuki Watanabe, Youiti Ootuka, Nucl. Instrum. Methods Phys. Res. A **386**, 439 (1997).
32. K. Zuber, Phys. Lett. B **519**, 1 (2001), arXiv: nucl-ex/0105018.
33. F. Frisk, I. Hamamoto, X.Z. Zhang, Phys. Rev. C **52**, 2468 (1995).
34. P. Sarriguren, E. Moya de Guerra, A. Escuderos, A.C. Carrizo, Nucl. Phys. A **635**, 55 (1998); P. Sarriguren, E. Moya de Guerra, A. Escuderos, Nucl. Phys. A **658**, 13 (1999); **691**, 631 (2001).
35. E. Nácher *et al.*, Phys. Rev. Lett. **92**, 232501 (2004).
36. L. Pacearescu, A. Faessler, F. Simkovic, Phys. At. Nucl. **67**, 1210 (2004).
37. R. Álvarez-Rodríguez, P. Sarriguren, E. Moya de Guerra, L. Pacearescu, A. Faessler, F. Simkovic, Phys. Rev. C **70**, 064309 (2004).
38. R. Chandra, J. Singh, P.K. Rath, P.K. Raina, J.G. Hirsch, Eur. Phys. J. A **23**, 223 (2005).
39. J. Suhonen, O. Civitarese, Phys. Lett. B **497**, 221 (2001).
40. A. Shukla, P.K. Raina, R. Chandra, P.K. Rath, J.G. Hirsch, Eur. Phys. J. A **23**, 235 (2005).
41. N. Auerbach, D.C. Zheng, L. Zamick, B.A. Brown, Phys. Lett. B **304**, 17 (1993); N. Auerbach, G.F. Bertsch, B.A. Brown, L. Zhao, Nucl. Phys. A **56**, 190 (1993).
42. D. Troltenier, J.P. Draayer, J.G. Hirsch, Nucl. Phys. A **601**, 89 (1996).
43. M. Baranger, K. Kumar, Nucl. Phys. A **110**, 490 (1968).
44. B.M. Dixit, P.K. Rath, P.K. Raina, Phys. Rev. C **65**, 034311 (2002); **67**, 059901 (2003)(E).
45. O. Civitarese, J. Suhonen, Phys. Rev. C **47**, 2410 (1993).
46. A. Bohr, B.R. Mottelson, *Nuclear Structure*, Vol. **I** (World Scientific, Singapore, 1998).
47. O. Castaños, J.G. Hirsch, O. Civitarese, P.O. Hess, Nucl. Phys. A **571**, 276 (1994).
48. J.G. Hirsch, O. Castaños, P.O. Hess, O. Civitarese, Phys. Rev. C **51**, 2252 (1995).
49. V.E. Ceron, J.G. Hirsch, Phys. Lett. B **471**, 1 (1999).
50. N. Onishi, S. Yoshida, Nucl. Phys. A **260**, 226 (1966).
51. J.D. Vergados, T.T.S. Kuo, Phys. Lett. B **35**, 93 (1971).
52. S.K. Khosa, P.N. Tripathi, S.K. Sharma, Phys. Lett. B **119**, 257 (1982); P.N. Tripathi, S.K. Sharma, S.K. Khosa, Phys. Rev. C **29**, 1951 (1984); S.K. Sharma, P.N. Tripathi, S.K. Khosa, Phys. Rev. C **38**, 2935 (1988).
53. G.M. Heestand, R.R. Borchers, B. Herskind, L. Grodzins, R. Kalish, D.E. Murnick, Nucl. Phys. A **133**, 310 (1969).
54. W. Greiner, Nucl. Phys. **80**, 417 (1966).
55. A. Arima, Nucl. Phys. A **354**, 19 (1981).
56. M. Sakai, At. Data Nucl. Data Tables **31**, 400 (1984).
57. S. Raman, C.H. Malarkey, W.T. Milner, C.W. Nestor jr., P.H. Stelson, At. Data Nucl. Data Tables **36**, 1 (1987).
58. P. Raghavan, Atomic Data Nucl. Data Tables **42**, 189 (1989).
59. O.A. Rumyantsev, M.G. Urin, Phys. Lett. B **443**, 51 (1998).
60. M. Hirsch, M. Muto, T. Oda, H.V. Klapdor-Kleingrothaus, Z. Phys. A **347**, 151 (1994).
61. A. Staudt, K. Muto, H.V. Klapdor-Kleingrothaus, Phys. Lett. B **268**, 312 (1991).
62. J. Suhonen, Phys. Rev. C **48**, 574 (1993).
63. J. Toivanen, J. Suhonen, Phys. Rev. C **55**, 2314 (1997).
64. O. Civitarese, J. Suhonen, Phys. Rev. C **58**, 1535 (1998).
65. S. Stoica, H.V. Klapdor-Kleingrothaus, Eur. Phys. J. A **17**, 529 (2003).